Examination Number:

Set:

Shore

Year 12
Trial HSC Examination
August 2015

Mathematics

General Instructions

- Reading time − 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11−16, show relevant mathematical reasoning and/or calculations
- Start each of Questions 11−16 in a new writing booklet
- Write your examination number on the front cover of each booklet to be handed in
- If you do not attempt a question, submit a blank booklet marked with your examination number and "N/A" on the front cover

Total marks − 100

Section I Pages 2–5

10 marks

- Attempt questions 1–10
- Allow about 15 minutes for this section

Section II Pages 6–14

90 marks

- Attempt questions 11–16
- Allow about 2 hours and 45 minutes for this section

Note: Any time you have remaining should be spent revising your answers.

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

Section I

10 marks
Attempt Questions 1−10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1−10.

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1 The value of $\dfrac{6.23+0.67}{\sqrt{8.21-2.15}}$ is closest to:

(A) 1

(B) 2

(C) 3

(D) 5

2 What is the primitive of $\dfrac{3}{x}-\sin x$?

(A) $-\dfrac{3}{x^2}+\cos x+c$

(B) $-\dfrac{3}{x^2}-\cos x+c$

(C) $3\ln x-\cos x+c$

(D) $3\ln x+\cos x+c$

3 What are the values of x for which $|4-3x|<13$?

(A) $x<-\dfrac{17}{3}$ or $x>3$

(B) $-\dfrac{17}{3}<x<3$

(C) $x<-3$ or $x>\dfrac{17}{3}$

(D) $-3<x<\dfrac{17}{3}$

4 Which of the following represents $2x^2 - 5x - 12$ in fully factorised form?

(A) $(2x-4)(x+3)$

(B) $(2x+4)(x-3)$

(C) $(2x+3)(x-4)$

(D) $(2x-3)(x+4)$

5 What are the values of p and q given $\left(2+\sqrt{3}\right)\left(1+\sqrt{12}\right) = p + q\sqrt{3}$?

(A) $p = 8$ and $q = 5$

(B) $p = 2$ and $q = 11$

(C) $p = 8$ and $q = 11$

(D) $p = 2$ and $q = 5$

6 The line $3x - ky = 5$ passes through the point (3,1). What is the value of k ?

(A) $-\frac{2}{3}$

(B) -7

(C) -4

(D) 4

7 A parabola has the equation $x^2 = 8(y+2)$. The coordinates of its vertex (V) and focus (S) are:

(A) $V(0,2)$ and $S(0,0)$

(B) $V(0,-2)$ and $S(0,0)$

(C) $V(0,2)$ and $S(0,-4)$

(D) $V(0,-2)$ and $S(0,-4)$

8 The semi-circle $y = \sqrt{9-x^2}$ is rotated about the *x*-axis. Which of the following expressions is correct for the volume of the solid of revolution?

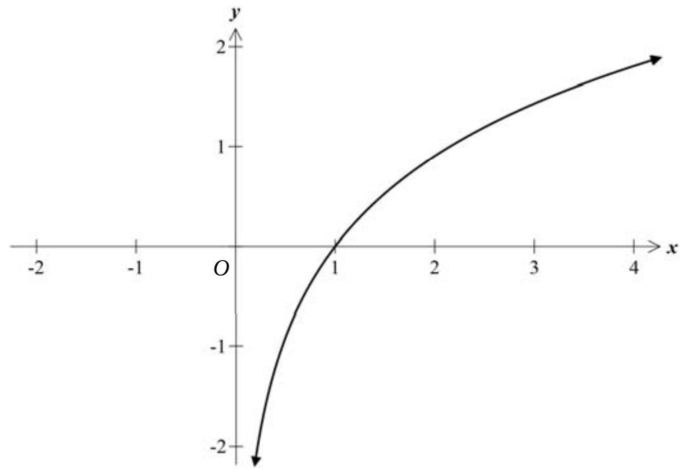
(A) $V = \pi\int_0^3 \left(9-x^2\right)dx$

(B) $V = 2\pi\int_0^3 \left(9-x^2\right)dx$

(C) $V = \pi\int_0^3 \left(9-y^2\right)dy$

(D) $V = 2\pi\int_0^3 \left(9-y^2\right)dy$

9 The graph below has which of the following properties?



(A) $f'(x) < 0$ and $f''(x) > 0$

(B) $f'(x) < 0$ and $f''(x) < 0$

(C) $f'(x) > 0$ and $f''(x) < 0$

(D) $f'(x) > 0$ and $f''(x) > 0$

10 What are the solutions of $2\sin x + 1 = 0$ for $0 \le x \le 2\pi$?

(A) $\frac{\pi}{3}$ and $\frac{2\pi}{3}$

(B) $\frac{4\pi}{3}$ and $\frac{5\pi}{3}$

(C) $\frac{\pi}{6}$ and $\frac{5\pi}{6}$

(D) $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$

Section II

90 marks
Attempt Questions 11−16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11−16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPERATE writing booklet.

(a) Solve $\frac{x-3}{4} = 3 - \frac{4x-3}{2}$. **2**

(b) Express $0.\dot{1}\dot{6}$ as an infinite geometric series and hence find its value as a fraction in simplest form. **2**

(c) Show that $\frac{\sqrt{5}}{\sqrt{5}-1} + \frac{\sqrt{5}}{\sqrt{5}+1}$ is rational. **2**

(d) Differentiate with respect to x

(i) $x^2 \log_e x$. **2**

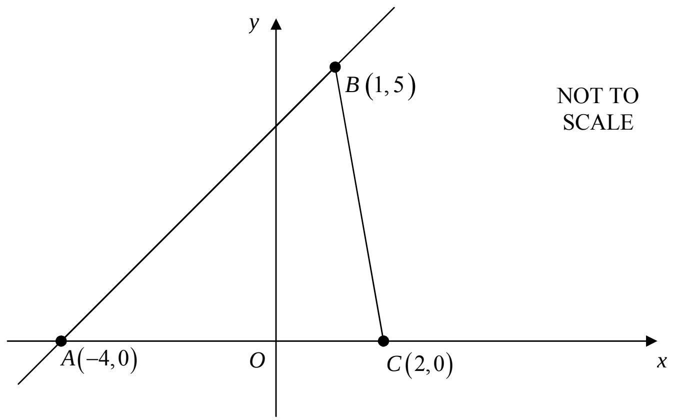
(ii) $e^{\sin^2 x}$. **2**

(iii) $\frac{x}{x^2+1}$. **2**

(e) Sketch the graph of $y = e^{-x} - 2$ clearly showing any asymptotes and intercepts on the x and y axes. **3**

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a)



(i) Show that the equation of the line passing through $A(-4, 0)$ and $B(1,5)$ is $x - y + 4 = 0$. **2**

(ii) Find the perpendicular distance of $C(2,0)$ to the line AB, in exact form. **1**

(iii) Hence, evaluate the area of triangle ABC. **2**

(iv) Find the coordinates of the centre and the radius of the circle $x^2 - 4x + y^2 = 21$. **2**

(v) Show that the line AB cuts the circle in (iv) at 2 points. Do not find the coordinates of these points. **2**

(b) Find the volume when the curve $y = \log_e x$ is rotated through 360° about the y-axis between $y = 1$ and $y = 4$. Write your answer in exact form. **3**

(c) Find the exact value of $\int_0^1 \sin\left(\pi x - \frac{\pi}{3}\right) dx$. **3**

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) In the first week of the snow season 5 cm of snow falls. In each following week the snowfalls increase by 2 cm, so in the second week 7 cm of snow falls, in the third week 9 cm of snow falls. This continues to the middle week and, from then on, the weekly snowfall decreases by 2 cm per week, the season lasting 21 weeks.

(i) How much snow falls in the 11^{th} (middle) week? **1**

(ii) What is the total snowfall for the whole season? **3**

(b) $\log_m p = 1.75$ and $\log_m q = 2.25$. Find:

(i) $\log_m \frac{q}{p}$. **1**

(ii) $\sqrt[5]{pq^2}$ in terms of m. **2**

(c) Show that $\dfrac{\sec\theta - \sec\theta\cos^4\theta}{1+\cos^2\theta} = \sin\theta\tan\theta$. **3**

(d) James borrows \$650 000 to buy a house and land package at 9% per annum reducible monthly interest over a period of 15 years. He agrees to repay the loan in equal monthly instalments of \$$M$.

(i) Show that \$$A_2$, the amount owing at the end of the second month, just after the second instalment of \$$M$ has been repaid, is given by **2**

$$A_2 = 650\ 000R^2 - M(1+R) \quad \text{where } R = \left(1 + \frac{9}{1200}\right).$$

(ii) Find the value of M, correct to the nearest cent. **3**

Question 14 (15 marks) Use a SEPARATE writing booklet.

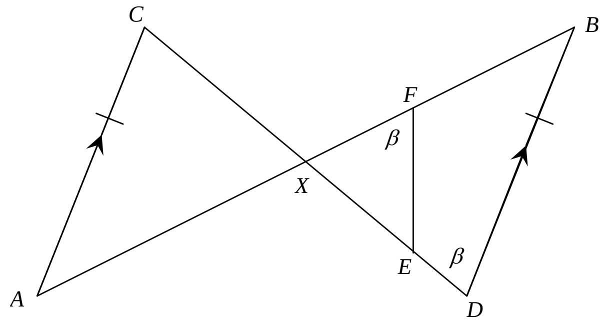
(a) The quadratic expression $2x^2 - px + 2$ has roots α and β. Find the following:

(i) $\alpha + \beta$ **1**

(ii) $\alpha\beta$ **1**

(iii) $\alpha^3\beta + \alpha\beta^3$ **2**

(b) In the diagram below, AC is parallel and equal to DB, $\angle XDB = \angle XFE = \beta$.



(i) Prove that ΔAXC is congruent to ΔBXD. **2**

(ii) Prove that ΔFXE is similar to ΔDXB. **2**

(iii) Hence or otherwise prove $\dfrac{XF}{XC} = \dfrac{EF}{AC}$. **2**

Question 14 continues on page 10

(c) Mark and Sam play poker. On each hand played, Mark has an 80% chance of winning.

(i) If they play two hands, what is the probability that Mark wins both hands? **1**

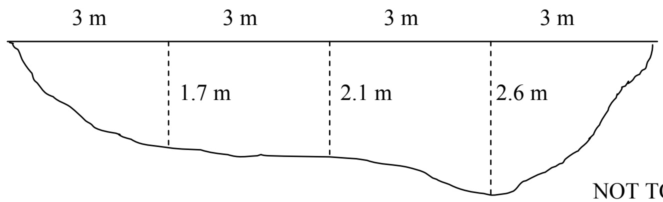
(ii) If they play two hands, what is the probability that Sam wins at least one hand? **1**

(iii) If they play *n* hands, what is the probability, in terms of *n*, that Sam wins at least one hand? **1**

(iv) What is the minimum number of hands that must be played so that Sam is 95% certain of winning at least one hand? **2**

Question 15 (15 marks) Use a SEPARATE writing booklet.

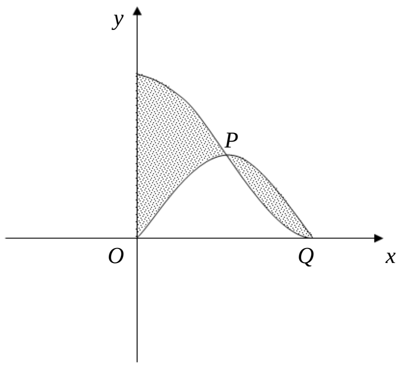
(a)



NOT TO SCALE

The diagram shows the cross-section of a 12 metre wide pond. The depths are taken every 3 metres. Use Simpson’s rule with five function values to find an approximate value for the area of the cross-section. **2**

(b)



NOT TO SCALE

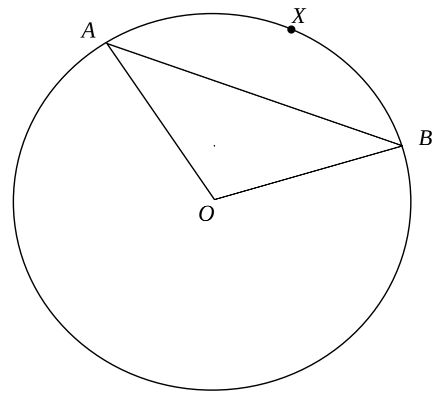
The graphs of $y = \sin x$ and $y = 1 + \cos x$ intersect at points P and Q, where Q is $(\pi, 0)$.

(i) Show that the point P is $\left(\frac{\pi}{2}, 1\right)$. **1**

(ii) Calculate the total area of the two shaded regions. **3**

Question 15 continues on page 12

(c)



NOT TO SCALE

(i) The area of the sector OAB is $\frac{100\pi}{3}\text{cm}^2$. Given that the radius of the sector is 10 cm, find the size of $\angle AOB$ in exact form. **1**

(ii) Hence, or otherwise, find the area of the minor segment AXB in exact form. **2**

(d) Consider the function $y = 1 - 3x + x^3$.

(i) Find the coordinates of the stationary points and determine their nature. **3**

(ii) Find the coordinates of any points of inflexion. **2**

(iii) Draw a sketch of the curve $y = 1 - 3x + x^3$ clearly showing all its essential features. **1**

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) An ambulance is delivering a patient to the hospital who is unconscious from a drug overdose. The doctor on duty does not know how much of the drug the unconscious patient has taken.

The rate of change of the concentration of the drug in the blood is proportional to the concentration, i.e. $\frac{dC}{dt} = kC$, where C mg/L is the concentration of the drug in the blood, t hours after the drug was initially taken.

(i) Show that $C = C_0 e^{kt}$ is a solution to $\frac{dC}{dt} = kC$ **1**

(ii) Three hours after the patient took the overdose, the blood concentration of the drug was 2.45 mg/L. Half an hour later the concentration was 1.84 mg/L. Show that the initial concentration of the drug in the patient's blood is 13.65 mg/L, correct to two decimal places. **3**

(iii) If the doctor on duty does not give the patient any further medication, when will the drug concentration fall below the critical value of 0.5 mg/L? Answer correct to one decimal place. **1**

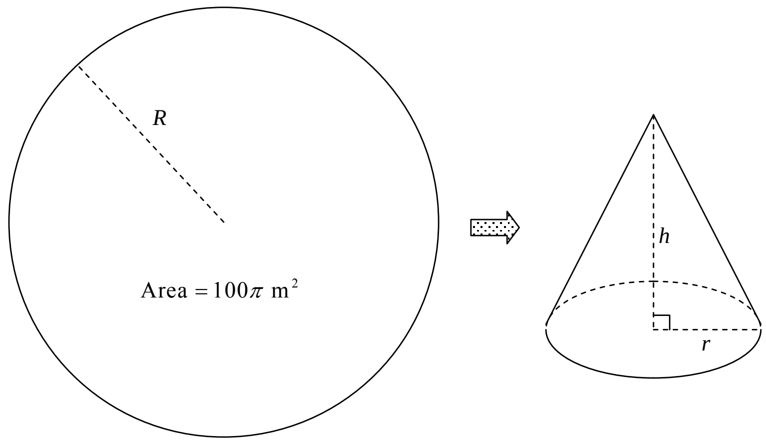
(b) Two particles moving in a straight line are initially at the origin. The velocity of one particle is $\frac{2}{\pi}$ m s^{-1} and the velocity of the other particle at t seconds is given by $v = -2\cos t$ m s^{-1}.

(i) Determine equations that give the displacements, x_1 and x_2 metres, of the particles from the origin at time t seconds. **2**

(ii) Show graphically that the particles will never meet again. **2**

Question 16 continues on page 14

(c) From a circular disc of metal whose area is $100\pi\ \text{m}^2$ a sector is cut and used to make a right cone. The radius of the disc is R metres.



(i) If the right cone has base radius r metres and height h metres, show that the volume of the cone is given by **2**

$$V = \frac{\pi r^2 \sqrt{100 - r^2}}{3}.$$

(iii) Show that the maximum volume of the cone occurs when $r = \sqrt{\frac{200}{3}}$. **4**

End of paper

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STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \; x \neq 0, \; \text{if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x \;, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note $\ln x = \log_e x, \quad x > 0$

(1) $2.8029334 \doteqdot 3$ $\therefore$ C

(2) $\int \left(\frac{3}{x} - \sin x\right) dx = 3\ln x + \cos x + c$ $\therefore$ D

(3) $|4 - 3x| < 13$

$\therefore -13 < 4 - 3x < 13$

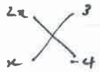
$-17 < -3x < 9$

$\therefore 17 > 3x > -9$

$(\div 3)$ $-3 < x < \frac{17}{3}$ $\therefore$ D

(4) $2x^2 - 5x - 12$

$= (2x+3)(x-4)$



$\therefore$ C

(5) LHS $= (2+\sqrt{3})(1+\sqrt{12})$

$= 2 + 2\sqrt{12} + \sqrt{3} + \sqrt{36}$

$= 2 + 2.2\sqrt{3} + \sqrt{3} + 6$

$= 8 + 5\sqrt{3}$

$= p + q\sqrt{3} \rightarrow p = 8, q = 5$ $\therefore$ A

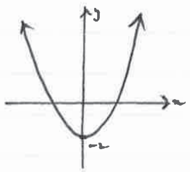
(6) Subst $(3,1)$ into $3x - ky = 5$

$\therefore 9 - k = 5$

$k = 4$ $\therefore$ D

(7) $x^2 = 8(y+2)$

$4a = 8 \rightarrow a = 2$



$V(0,-2)$

$\therefore S(0,0)$

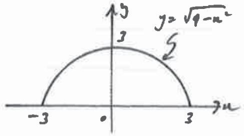
$\therefore$ B

(8) $y = \sqrt{9-x^2}$

$\therefore y^2 = 9 - x^2$

$V = \pi \int_a^b y^2\,dx$

$= 2\pi \int_0^3 (9-x^2)\,dx$ by symmetry



$\therefore$ B

(9) Curve is increasing $\rightarrow f'(x) > 0$

curve is concave down $\rightarrow f''(x) < 0$

$\therefore$ C

(10) $2\sin x + 1 = 0$ $0 \le x \le 2\pi$

$\sin x = -\frac{1}{2}$

B.A $= \frac{\pi}{6}$

$\sin x < 0$ in 3rd, 4th quads

$\therefore x = \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$

$= \frac{7\pi}{6}, \frac{11\pi}{6}$

$\therefore$ D

(11) (a) $\frac{x-3}{4} = 3 - \frac{4x-3}{2}$

$(\times 8)$ $2(x-3) = 24 - 4(4x-3)$

$2x - 6 = 24 - 16x + 12$

$18x = 42$

$x = \frac{42}{18} = 2\frac{1}{3}$

(b) $0 \cdot \dot{1}\dot{6} = 0 \cdot 161616 \ldots$

$= \frac{16}{10^2} + \frac{16}{10^4} + \ldots$

$= S_\infty$ of GP, $a = \frac{16}{10^2}$, $r = \frac{1}{10^2}$

$= \frac{a}{1-r}$

$= \frac{16/100}{1 - \frac{1}{100}}$

$= \frac{16/100}{99/100}$

$= \frac{16}{99}$

(c) Expn $= \frac{\sqrt{5}(\sqrt{5}+1) + \sqrt{5}(\sqrt{5}-1)}{(\sqrt{5}-1)(\sqrt{5}+1)}$

$= \frac{5 + \sqrt{5} + 5 - \sqrt{5}}{5-1}$

$= \frac{5}{2}$ (rational)

(d) (i) $y = x^2 \ln x$

$y' = x^2 \cdot \frac{1}{x} + \ln x \cdot 2x$

$= x + 2x \ln x$

$= x(1 + 2\ln x)$

(ii) $y = e^{\sin^2 x}$

$y' = e^{\sin^2 x} \cdot 2\sin x \cos x$

$= 2\sin x \cos x \, e^{\sin^2 x}$

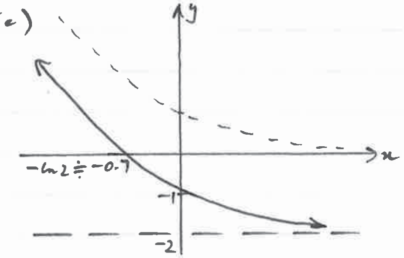
(iii) $y = \frac{x}{x^2+1}$

$y' = \frac{(x^2+1).1 - x.2x}{(x^2+1)^2}$

$= \frac{x^2 + 1 - 2x^2}{(x^2+1)^2}$

$= \frac{1-x^2}{(x^2+1)^2}$

(e)



$y = e^{-x} - 2$

$x = 0 \rightarrow y = e^0 - 2 = 1 - 2 = -1$

$y = 0 \rightarrow e^{-x} = 2$

$-x = \log_e 2$

$\therefore x = -\ln 2 \doteqdot -0.7$

(12) (a)(i) $m_{AB} = \frac{5-0}{1+4} = 1$

$y - 0 = 1(x+4)$

$y = x + 4$

$\therefore x - y + 4 = 0$

(ii) $p = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2+B^2}}$ $C(2,0)$

$= \frac{|1(2) + (-1)(0) + 4|}{\sqrt{1+1}}$

$= \frac{6}{\sqrt{2}}$ or $3\sqrt{2}$ u

(iii) $AB = \sqrt{(1+4)^2 + (5-0)^2}$

$= \sqrt{25+25}$

$= 5\sqrt{2}$ u

$\therefore$ Area $\Delta ABC = \frac{1}{2} b h$

$= \frac{1}{2} \cdot 5\sqrt{2} \cdot \frac{6}{\sqrt{2}}$

$= 15$ u^2

(iv) $(x^2 - 4x + \underline{4}) + y^2 = 21 + \underline{4}$

$(x-2)^2 + y^2 = 25$

Centre is $(2,0)$, $r=5$

(v) $p = 3\sqrt{2} \doteqdot 4.2\ u$ (from ii)

$r = 5\ u$ (from iv)

Since $r > p$, circle will cut line AB in 2 points

OR

$x - y + 4 = 0 \quad \ldots(1)$

$x^2 - 4x + y^2 = 21 \quad \ldots(2)$

From (1), $y = x + 4$

Subst in (2):

$x^2 - 4x + (x+4)^2 = 21$

$x^2 - 4x + x^2 + 8x + 16 = 21$

$2x^2 + 4x - 5 = 0$

$x = \dfrac{-4 \pm \sqrt{16 - 4(2)(-5)}}{4}$

$= \dfrac{-4 \pm \sqrt{56}}{4}$

$\therefore$ 2 pts of intersection

OR $\Delta = b^2 - 4ac$

$= 16 - 4(2)(-5)$

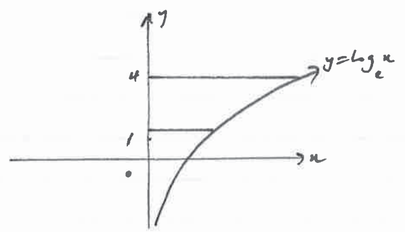
$= 56$

> 0

$\therefore$ 2 roots

$\therefore$ 2 pts of intersection

(b)



$y = \log_e x \rightarrow x = e^y$

$x^2 = (e^y)^2 = e^{2y}$

$V = \pi \int_c^d x^2\, dy$

$= \pi \int_1^4 e^{2y}\, dy$

$= \frac{\pi}{2}\left[e^{2y}\right]_1^4$

$= \frac{\pi}{2}\left(e^8 - e^2\right)\ u^3.$

(c) $\int_0^1 \sin\left(\pi x - \frac{\pi}{3}\right) dx$

$= -\frac{1}{\pi}\left[\cos\left(\pi x - \frac{\pi}{3}\right)\right]_0^1$

$= -\frac{1}{\pi}\left[\cos\left(\pi - \frac{\pi}{3}\right) - \cos\left(-\frac{\pi}{3}\right)\right]$

$= -\frac{1}{\pi}\left[-\cos\frac{\pi}{3} - \cos\frac{\pi}{3}\right]$

$= -\frac{1}{\pi}\cdot\left[-\frac{1}{2} - \frac{1}{2}\right]$

$= -\frac{1}{\pi}\cdot -1$

$= \frac{1}{\pi}$

(13) (a) $5, 7, 9, \ldots$

(i) AP, $a=5$, $d=2$, $n=11$

$T_n = a + (n-1)d$

$T_{11} = 5 + 10 \times 2$

$= 25$

$\therefore$ 25cm of snow in 11^{th} week

(ii) $\overset{T_1}{5}, 7, 9, \ldots, \overset{T_{11}}{25}, 23, 21, \ldots$

$S_n = \frac{n}{2}\left(2a + [n-1]d\right)$

$\therefore$ Total $= \frac{11}{2}(10 + 10 \times 2) + \frac{10}{2}\left(2 \times 23 + 9 \times (-2)\right)$

$= \frac{11}{2} \times 30 + 5 \times 28$

$= 305$

$\therefore$ 305 cm of snow over 21 week season

[or Total $= 2 \times S_{10} + T_{11}$]

(b) $\log_m p = 1.75 \qquad \log_m q = 2.25$

(i) $\log_m \frac{q}{p} = \log_m q - \log_m p$

$= 2.25 - 1.75 = \underline{0.5}$

(ii) $\log_m p = 1.75 \rightarrow p = m^{1.75}$

$\log_m q = 2.25 \rightarrow q = m^{2.25}$

$(pq^2)^{1/5} = \left(m^{1.75} \times m^{4.5}\right)^{1/5}$

$= \left(m^{6.25}\right)^{1/5}$

$= \underline{m^{1.25}}$

(c) LHS $= \dfrac{\sec\theta - \sec\theta\cos^4\theta}{1 + \cos^2\theta}$

$= \dfrac{\sec\theta\left(1 - \cos^4\theta\right)}{1 + \cos^2\theta}$

$= \dfrac{\sec\theta\left(1 + \cos^2\theta\right)\left(1 - \cos^2\theta\right)}{1 + \cos^2\theta}$

$= \sec\theta\left(1 - \cos^2\theta\right)$

$= \sec\theta .\ \sin^2\theta$

$= \frac{1}{\cos\theta} .\ \sin\theta .\ \sin\theta$

$= \sin\theta .\ \frac{\sin\theta}{\cos\theta}$

$= \sin\theta \tan\theta$ = RHS

(d) Let A_n = amt owing after n months

M = monthly repayment

9% p.a $= \frac{9}{12} = 0.75\%$ p.m

(i) $A_1 = 650\,000 + 650\,000 \times 0.0075 - M$

$= 650\,000(1.0075) - M$

$A_2 = A_1(1.0075) - M$

$= \left(650\,000(1.0075) - M\right)1.0075 - M$

$= 650\,000(1.0075)^2 - 1.0075M - M$

$= 650\,000(1.0075)^2 - M(1 + 1.0075)$

$= 650\,000R^2 - M(1 + R)$

where $R = 1.0075 = 1 + \frac{9}{1200}$

(ii) $A_{180} = 650\,000(1.0075)^{180} - M\left(1 + 1.0075 + \ldots + 1.0075^{179}\right)$

We require $A_{180} = 0$

$\therefore M = \dfrac{650\,000\,(1.0075)^{180}}{1 + 1.0075 + \ldots + 1.0075^{179}}$

↖ S_{180} of GP, $a=1$, $r=1.0075$

$= 650000(1.0075)^{180} \div \dfrac{1\left(1.0075^{180} - 1\right)}{1.0075 - 1}$

$= 650\,000(1.0075)^{180} \times \dfrac{0.0075}{1.0075^{180} - 1}$

$= 6592.772\ldots$

$\therefore\ = \underline{6592.77}$ (nearest cent)

(14) (a) $2x^2 - px + 2$

(i) $\alpha + \beta = \frac{-b}{a} = \frac{p}{2}$

(ii) $\alpha\beta = \frac{c}{a} = \frac{2}{2} = 1$

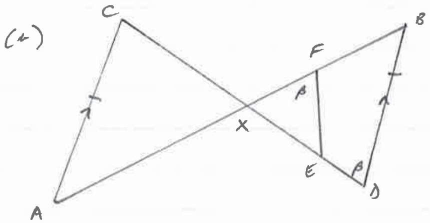
(iii) $\alpha^3\beta + \alpha\beta^3 = \alpha\beta\left(\alpha^2 + \beta^2\right)$

$= \alpha\beta\left[(\alpha+\beta)^2 - 2\alpha\beta\right]$

$= 1\left[\left(\frac{p}{2}\right)^2 - 2 \times 1\right]$

$= \frac{p^2}{4} - 2$ or $\frac{p^2 - 8}{4}$

(b)



(i) In Δ's ACX, BXD

$\angle CAX = \angle DBX$ (alt ∠'s, AC ∥ DB)

$AC = BD$ (given)

$\angle ACX = \angle BDX$ (alt ∠'s, AC ∥ DB)

$\therefore \Delta ACX \equiv \Delta BXD$ (AAS)

(ii) In Δ's FXE, DXB

$\angle EXF$ is common

$\angle XFE = \angle XDB \; (=\beta)$

$\therefore \Delta FXE \;|||\; \Delta DXB$ (equiangular)

(iii) $\dfrac{XF}{XD} = \dfrac{EF}{BD}$ (corresp sides of similar Δ's)

But $XD = XC$ (corresp sides of cong Δ's)

and $BD = AC$ (corresp sides of cong Δ's)

$\therefore \dfrac{XF}{XC} = \dfrac{EF}{AC}$

(c) $P(M_W) = 0.8$

(i) $P(M_W M_W) = 0.8 \times 0.8$

$= 0.64$

(ii) P(Sam wins at least 1 hand)

$= 1 - P$(Sam loses both hands)

$= 1 - P(M_W M_W)$

$= 1 - 0.64$

$= 0.36$

(iii) P(Sam wins at least 1 hand)

$= 1 - P(M_W M_W \ldots M_W)$ ← n terms

$= 1 - (0.8)^n$

(iv) We require $1 - 0.8^n \geq 0.95$

$0.05 \geq 0.8^n$

$0.8^n \leq 0.05$

(Take $\log_{10}$ of b.s) $n \log_{10} 0.8 \leq \log_{10} 0.05$

($\div \log_{10} 0.8$) $n \geq \dfrac{\log 0.05}{\log 0.8}$

N.B. reversal of sign, because we are dividing b.s. by a negative

$\therefore \; n \geq 13.425\ldots$

$\therefore$ minimum number of hands is 14

(15)(a) $A \doteqdot \frac{h}{3}\left[y_0 + y_4 + 4(y_1 + y_3) + 2y_2\right]$

$= \frac{3}{3}\left[0 + 0 + 4(1.7 + 2.6) + 2(2.1)\right]$

$= 21.4 \text{ m}^2$

(b) (i) Subst $\left(\frac{\pi}{2}, 1\right)$ into $y = \sin x$

$\therefore \; 1 = \sin \frac{\pi}{2}$ (T)

$\therefore$ pt lies on $y = \sin x$

Subst $\left(\frac{\pi}{2}, 1\right)$ into $y = 1 + \cos x$

$\therefore \; 1 = 1 + \cos \frac{\pi}{2}$

$1 = 1 + 0$ (T)

$\therefore$ pt lies on $y = 1 + \cos x$.

(ii) $A = \displaystyle\int_0^{\frac{\pi}{2}} (1 + \cos x - \sin x)\, dx + \int_{\frac{\pi}{2}}^{\pi} (\sin x - [1 + \cos x])\, dx$

$= \left[x + \sin x + \cos x\right]_0^{\pi/2} + \left[-\cos x - x - \sin x\right]_{\frac{\pi}{2}}^{\pi}$

$= \left(\frac{\pi}{2} + \sin\frac{\pi}{2} + \cos\frac{\pi}{2}\right) - (0 + \sin 0 + \cos 0) + \left(-\cos\pi - \pi - \sin\pi\right) - \left(-\cos\frac{\pi}{2} - \frac{\pi}{2} - \sin\frac{\pi}{2}\right)$

$= \frac{\pi}{2} + 1 - 1 + 1 - \pi + \frac{\pi}{2} + 1$

$= 2 \text{ u}^2$.

(c)(i) $A = \frac{1}{2} r^2 \theta$

$\therefore \dfrac{100\pi}{3} = \frac{1}{2} \cdot 100 \cdot \theta$

$\therefore \theta = \dfrac{2\pi}{3}$

(ii) Area of minor segment

$= \frac{1}{2} r^2 (\theta - \sin\theta)$

$= \frac{1}{2} \cdot 100 \left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right)$

$= \frac{1}{2} \cdot 100 \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$

$= 50 \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$

$= \left(\dfrac{100\pi}{3} - 25\sqrt{3}\right) \text{cm}^2$

(d) $y = 1 - 3x + x^3$

$y' = -3 + 3x^2$

$y'' = 6x$

(i) For a stat pt, $y' = 0$

$\therefore \; -3 + 3x^2 = 0$

$x^2 = 1$

$\therefore \; x = \pm 1$

When $x = 1$, $y = 1 - 3 + 1 = -1$

$x = -1$, $y = 1 + 3 - 1 = 3$

Test $(1, -1)$ When $x = 1$, $y'' = 6 > 0$ ↺

$\therefore$ min t.p.

Test $(-1, 3)$ When $x = -1$, $y'' = -6$ ↻

$\therefore$ max. t.p.

(ii) For a P.O.I., $y'' = 0$

$6x = 0$

$x = 0$

When $x = 0$, $y = 1 - 0 + 0 = 1$

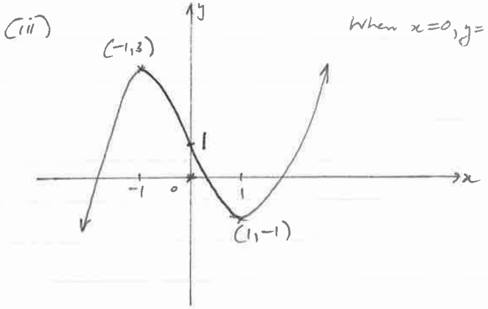
Test $(0, 1)$

x	0^-	0	0^+
y''	$-$	0	$+$

$\therefore$ concavity change

$\therefore (0, 1)$ is a P.O.I.

(iii)



When $x = 0$, $y = 1$

(16) (i) $C = C_0 e^{kt}$

$\dfrac{dC}{dt} = C_0 e^{kt} \cdot k$

$= k(C_0 e^{kt})$

$= kC$

(ii) $t = 3$, $C = 2.45 \rightarrow 2.45 = C_0 e^{3k} \; \cdots (1)$

$t = 3.5$, $C = 1.84 \rightarrow 1.84 = C_0 e^{3.5k} \; \cdots (2)$

$(2) \div (1)$: $\dfrac{1.84}{2.45} = \dfrac{C_0 e^{3.5k}}{C_0 e^{3k}}$

$\therefore \; e^{0.5k} = \dfrac{1.84}{2.45}$

$0.5k = \log_e\left(\dfrac{1.84}{2.45}\right)$

$k = \dfrac{\log\left(\frac{1.84}{2.45}\right)}{0.5}$

$= -0.572644905$

Subst in (1): $C_0 = \dfrac{2.45}{e^{3\times -0.572644905}}$

$= 13.65386832$

$= 13.65$ mg/L (2 d.p.)

(iii) $C = 13.65 e^{-0.572644905t}$

$C = 0.5$

$\rightarrow \quad 0.5 = 13.65 e^{-0.572644905t}$

$\frac{0.5}{13.65} = e^{-0.572644905t}$

$\therefore t = \frac{\log_e\left(\frac{0.5}{13.65}\right)}{-0.572644905}$

$= 5.7747\ldots$

$= 5.8$ (to 1 d.p.)

$\therefore$ after 5.8 hours.

(b)(i) $v_1 = \frac{2}{\pi}$

Integrating b.s. wrt t

$\therefore x_1 = \frac{2}{\pi} t + c$

$t=0, x_1 = 0 \rightarrow 0 = 0 + c \rightarrow c = 0$

$\therefore x_1 = \frac{2}{\pi} t$

$v_2 = -2\cos t$

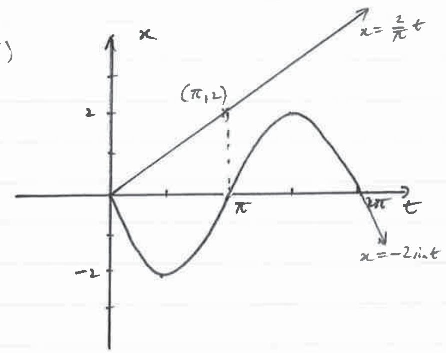
Integrating b.s. wrt t

$\therefore x_2 = -2\sin t + k$

$t=0, x_2 = 0 \rightarrow 0 = 0 + k \rightarrow k = 0$

$\therefore x_2 = -2\sin t$.

(ii)



The graphs don't intersect again.

$\therefore$ particles never meet again.

(c)(i) $A = \pi R^2$

$\therefore 100\pi = \pi R^2$

$\therefore R^2 = 100$

$R = \pm 10$

Since $R > 0$, $R = 10$

Now, $h^2 + r^2 = 10^2$

$h^2 + r^2 = 100$

$h^2 = 100 - r^2$

$h = \sqrt{100 - r^2}$ (as $h > 0$)

$V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi r^2 \sqrt{100 - r^2}$

(ii) $\frac{dV}{dr} = \frac{1}{3}\pi \left[r^2 \cdot \frac{1}{2}(100 - r^2)^{-\frac{1}{2}} \cdot -2r + \sqrt{100 - r^2} \cdot 2r \right]$

$= \frac{1}{3}\pi \left[\frac{-r^3}{\sqrt{100 - r^2}} + 2r\sqrt{100 - r^2} \right]$

$= \frac{1}{3}\pi \frac{-r^3 + 2r(100 - r^2)}{\sqrt{100 - r^2}}$

$= \frac{1}{3}\pi \frac{-r^3 + 200r - 2r^3}{\sqrt{100 - r^2}}$

$= \frac{1}{3}\pi \frac{200r - 3r^3}{\sqrt{100 - r^2}}$

For a stat pt, $\frac{dV}{dr} = 0$

$\therefore \frac{1}{3}\pi \frac{200r - 3r^2}{\sqrt{100 - r^2}} = 0$

$\therefore r(200 - 3r^2) = 0$

$\therefore r = 0$ or $3r^2 = 200$

reject

$r^2 = \frac{200}{3}$

$r = \pm\sqrt{\frac{200}{3}}$

as $r > 0$, $r = \sqrt{\frac{200}{3}}$ ($\doteqdot 8.16$)

Test $r = \sqrt{\frac{200}{3}}$

r	8	$\sqrt{\frac{200}{3}}$	9
$\frac{dV}{dr}$	+	0	−

$\therefore V$ is a max when $r = \sqrt{\frac{200}{3}}$